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A note on quantum entropy inequalities and channel capacities

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Abstract
Quantum entropy inequalities are studied. Some quantum entropy inequalities are obtained by several methods. For an entanglement breaking channel, we show that the entanglement-assisted classical capacity is upper bounded by $\log d$. A relationship between entanglement-assisted and one-shot unassisted capacities is obtained. This relationship shows the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the one-shot unassisted classical capacity.

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1. Introduction

Quantum information theory has been attracting a great deal of interest. Several capacities of quantum channels have been proposed and studied, such as the Holevo–Schumacher–Westmoreland channel capacity [1, 2] and the recently proposed entanglement-assisted classical capacity [3, 4] and adaptive classical capacity [5]. In studying capacities of quantum channels, the quantum entropy inequalities are very important. In [6–11], a survey of quantum entropy inequalities is presented. Some of these quantum inequalities are independent but equivalent, i.e. they are necessary and sufficient conditions to each other [8]. In some cases, the results can be obtained much easily from one quantum entropy inequality than from others. So, all of these inequalities are necessary. It will be better if we can find more inequalities. In this paper, we study some of these quantum entropy inequalities and find their applications in channel capacities.

The additivity of classical capacity of quantum channels is one of the fundamental problems in studying the quantum information theory. The additivity of classical capacity of several special channels is proved, such as unital qubit channels [12], depolarizing channels [13] and entanglement breaking channels [14]. By using the strong concavity of von Neumann entropy directly, we give a simple proof of the additivity of classical capacity of entanglement breaking channels.

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It is known that the classical capacity of quantum channels may be enhanced with prior entanglement such as the super-dense coding protocol [15]. A general theorem called entanglement-assisted classical capacity was proposed and proved recently concerning the classical capacity of quantum channels with the help of shared entanglement between the sender and the receiver [3, 4]. It can be expected that if the channel itself is entanglement breaking, its entanglement-assisted classical capacity has less advantage than other channels. We show in this paper that, for an entanglement breaking channel, the entanglement-assisted classical capacity is upper bounded by $\log d$ while generally we have an extra term χ.

A simple proof of the entanglement-assisted channel capacity was given by Holevo [16], who also found the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the unassisted classical capacity. We shall show further in this paper that the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the one-shot unassisted classical capacity. This result also eliminates one possible way in which one might find an example of non-additivity of the classical capacity.

2. Equivalent quantum entropy inequalities

There are four equivalent quantum entropy inequalities as reviewed by Ruskai [8]. In this section, we point out that we can add another equivalent entropy inequality.

First, let us introduce some definitions. The von Neumann entropy is defined as

$$S(\rho) \equiv -\mathrm{Tr}(\rho \log \rho) \tag{1}$$

where ρ is the density operator. The relative entropy is defined as

$$S(\rho\|\sigma) \equiv \mathrm{Tr}\, \rho(\log \rho - \log \sigma). \tag{2}$$

In a recent review, Ruskai listed the first four equivalent quantum entropy inequalities as follows (see [8] and the references therein):

(i) Monotonicity of relative entropy under completely positive, trace preserving maps:

$$S(\Phi(\rho)\|\Phi(\sigma)) \leqslant S(\rho\|\sigma). \tag{3}$$

(ii) Monotonicity of relative entropy under partial trace:

$$S(\rho_A\|\sigma_A) \leqslant S(\rho_{AC}\|\sigma_{AC}). \tag{4}$$

(iii) Strong subadditivity of von Neumann entropy I and II, where I and II are equivalent:

$$\begin{aligned} &(\mathrm{I}) \quad S(\rho_A) + S(\rho_B) \leqslant S(\rho_{AC}) + S(\rho_{BC});\\ &(\mathrm{II}) \quad S(\rho_{ABC}) + S(\rho_B) \leqslant S(\rho_{AB}) + S(\rho_{BC}). \end{aligned} \tag{5}$$

(iv) Joint convexity of relative entropy:

$$S\left(\sum_i p_i \rho^i \,\middle\|\, \sum_i p_i \sigma^i\right) \leqslant \sum_i p_i S(\rho^i\|\sigma^i). \tag{6}$$

(v) Actually, we can add another equivalent inequality, concavity of conditional entropy:

$$S\left(\sum_i p_i \rho_{AB}^i\right) - S\left(\sum_i p_i \rho_B^i\right) \geqslant \sum_i p_i\left[S\left(\rho_{AB}^i\right) - S\left(\rho_B^i\right)\right]. \tag{7}$$

The last inequality was deduced from (iv), joint convexity of relative entropy in [6]. Then it was used to deduce inequality (iii), strong subadditivity. So, inequality (v), the concavity of conditional entropy, is an equivalent inequality with the other four inequalities.

In the textbook of Nielsen and Chuang [6], inequality (v) is obtained from (iv), the joint convexity. Next, we show two other methods to obtain the concavity of conditional entropy. First, we use (ii), monotonicity of relative entropy under partial trace. Suppose $\rho_{AB} = \sum_i p_i \rho_{AB}^i$, so $\rho_B = \sum_i p_i \rho_B^i$. From inequality (ii), we have

$$S\big(\rho_B^i \big\| \rho_B\big) \leqslant S\big(\rho_{AB}^i \big\| \rho_{AB}\big). \tag{8}$$

So, the average of relative entropies has the inequality

$$\sum_i p_i S\big(\rho_B^i \big\| \rho_B\big) \leqslant \sum_i p_i S\big(\rho_{AB}^i \big\| \rho_{AB}\big). \tag{9}$$

From the definition of relative entropy, we obtain (v),

$$S(\rho_{AB}) - S(\rho_B) \geqslant \sum_i p_i\big[S\big(\rho_{AB}^i\big) - S\big(\rho_B^i\big)\big]. \tag{10}$$

Secondly, we also use the joint convexity to deduce (v), but by a different method. The joint convexity of relative entropy means

$$S\left(\sum_i p_i \rho_{AB}^i \,\middle\|\, \sum_i p_i \rho_B^i\right) \leqslant \sum_i p_i S\big(\rho_{AB}^i \big\| \rho_B^i\big). \tag{11}$$

By definition, we have

$$-S(\rho_{AB}) + S(\rho_B) \leqslant -\sum_i p_i\big[S\big(\rho_{AB}^i\big) - S\big(\rho_B^i\big)\big]. \tag{12}$$

This is exactly (v), the concavity of conditional entropy.

Since these five inequalities are equivalent, we can obtain any one of them from one of the other four inequalities. Recently, Bennett *et al* proposed and proved the entanglement-assisted channel capacity [3, 4]. Holevo subsequently gave a modified proof [16], and one of the simplifications is due to the replacement of strong subadditivity by concavity of conditional entropy, i.e., the fifth inequality was used directly in [16] rather than the third inequality used in [4], though they are equivalent.

3. Strong concavity of von Neumann entropy

In this section, we propose the following quantum entropy inequality: strong concavity of von Neumann entropy,

$$S\left(\sum_i p_i \rho_A^i \otimes \rho_B^i\right) \geqslant \max\left\{\sum_i p_i S\big(\rho_A^i\big) + S\left(\sum_i p_i \rho_B^i\right), \sum_i p_i S\big(\rho_B^i\big) + S\left(\sum_i p_i \rho_A^i\right)\right\}. \tag{13}$$

To prove this inequality, we need to show that both the following inequalities hold,

$$S\left(\sum_i p_i \rho_A^i \otimes \rho_B^i\right) \geqslant \sum_i p_i S\big(\rho_A^i\big) + S\left(\sum_i p_i \rho_B^i\right) \tag{14}$$

and

$$S\left(\sum_i p_i \rho_A^i \otimes \rho_B^i\right) \geqslant \sum_i p_i S\big(\rho_B^i\big) + S\left(\sum_i p_i \rho_A^i\right). \tag{15}$$

We denote $\rho_A = \sum_i p_i \rho_A^i$, $\rho_B = \sum_i p_i \rho_B^i$.

In the following, we present several methods to derive the strong concavity of quantum entropy.

(A) Due to (ii), monotonicity of relative entropy, we have

$$S(\rho_A^i\|\rho_A) \leqslant S\left(\rho_A^i\otimes\rho_B^i \,\middle\|\, \sum_i p_i\rho_A^i\otimes\rho_B^i\right). \tag{16}$$

Taking the average with probability p_i, we have

$$\sum_i p_i S(\rho_A^i\|\rho_A) \leqslant \sum_i p_i S\left(\rho_A^i\otimes\rho_B^i \,\middle\|\, \sum_j p_j\rho_A^j\otimes\rho_B^j\right). \tag{17}$$

So, we have

$$-\sum_i p_i S(\rho_A^i) + S(\rho_A) \leqslant -\sum_i p_i[S(\rho_A^i)+S(\rho_B^i)] + S\left(\sum_i p_i\rho_A^i\otimes\rho_B^i\right). \tag{18}$$

Thus we obtain the strong concavity.

(B) Due to (iv), joint convexity of relative entropy,

$$\begin{aligned} S\left(\sum_i p_i\rho_A^i\otimes\rho_B^i \,\middle\|\, \sum_i p_i\rho_A^i\right) &\leqslant \sum_i p_i S(\rho_A^i\otimes\rho_B^i\|\rho_A^i) \\ &= -\sum_i p_i S(\rho_B^i). \end{aligned} \tag{19}$$

We have the strong concavity of von Neumann entropy.

(C) Due to (v), concavity of conditional entropy, we have

$$\begin{aligned} S\left(\sum_i p_i\rho_A^i\otimes\rho_B^i\right) - S\left(\sum_i p_i\rho_A^i\right) &\geqslant \sum_i p_i[S(\rho_A^i\otimes\rho_B^i) - S(\rho_A^i)] \\ &= \sum_i p_i S(\rho_B^i). \end{aligned} \tag{20}$$

Then we arrive at the strong concavity. Since the situations for ρ_A and ρ_B are the same, we know that equation (13) holds. The strong concavity of von Neumann entropy can be obtained simply from some well-known quantum entropy inequalities. It should have been noted and applied implicitly or explicitly [6, 9]. We present it here since we will use it in the next sections to obtain some results.

4. Application of strong concavity in the channel capacity of an entanglement breaking channel

Recently, Shor proved the additivity of the classical capacity of an entanglement breaking quantum channel [14]. Both c-q (classical–quantum) and q-c (quantum–classical) channels are special cases of entanglement breaking channels. And the entanglement breaking channel can be expressed as a q-c-q channel. Other properties and conjectures for an entanglement breaking channel can be found in [17]. We next give a simple proof of the additivity of the channel capacity of an entanglement breaking channel by directly using the strong concavity inequality though there are no essential differences from Shor's original proof.

An entanglement breaking channel Φ means $(I \otimes \Phi)\rho_{AB}$ is always a separable state, which can be written as [18]

$$(I \otimes \Phi)\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i. \tag{21}$$

So we know $\Phi(\rho_B) = \sum_i p_i \rho_B^i$. Suppose $\sum_j q_j \rho_{AB}^j = \rho_{AB}$ are the optimal signal states for channel $\Psi \otimes \Phi$, where Ψ is an arbitrary quantum channel. The Holevo–Schumacher–Westmoreland channel capacity $\chi^*(\Psi \otimes \Phi)$ takes the following form

$$\begin{aligned}\chi^*(\Psi \otimes \Phi) &= \sum_j q_j S\left((\Psi \otimes \Phi)\big(\rho_{AB}^j\big) \,\middle\|\, (\Psi \otimes \Phi)\left(\sum_j q_j \rho_{AB}^j\right)\right) \\ &= -\sum_j q_j S\left(\sum_i p_{ji}\Psi\big(\rho_A^{ji}\big) \otimes \rho_B^{ji}\right) + S((\Psi \otimes \Phi)(\rho_{AB})).\end{aligned} \tag{22}$$

Then using the strong concavity inequality for the first term and subadditivity for the second term, we have

$$\begin{aligned}\chi^*(\Psi \otimes \Phi) &\leqslant -\sum_{ji} q_j p_{ji} S\big(\Psi\big(\rho_A^{ji}\big)\big) - \sum_j q_j S\left(\sum_i p_{ji}\rho_B^{ji}\right) + S(\Psi(\rho_A)) + S(\Phi(\rho_B)) \\ &= \sum_{ji} q_j p_{ji} S\big(\Psi\big(\rho_A^{ji}\big) \,\|\, \Psi(\rho_A)\big) + \sum_j q_j S\big(\Phi\big(\rho_B^j\big) \,\|\, \Phi(\rho_B)\big) \\ &\leqslant \chi^*(\Psi) + \chi^*(\Phi).\end{aligned} \tag{23}$$

Since the classical capacity of a quantum channel is strong additive, we know the capacity of an entanglement breaking channel is additive,

$$\chi^*(\Psi \otimes \Phi) = \chi^*(\Psi) + \chi^*(\Phi). \tag{24}$$

5. Application of strong concavity of von Neumann entropy in entanglement-assisted channel capacity for an entanglement breaking channel

Recently, Bennett *et al* [3, 4] (BSST theorem) proposed and proved the entanglement-assisted channel capacity in terms of quantum mutual information. Holevo [16] then gave a simple proof. The BSST theorem states that the classical capacity of the entanglement-assisted channel is written in the form

$$C_E(\Phi) = \max\nolimits_{\rho_A \in \mathcal{H}_{in}} S(\rho_A) + S(\Phi(\rho_A)) - S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)) \tag{25}$$

where $|\Psi_{AB}\rangle$ is a purification of ρ_A.

Holevo [16] pointed out that there is a relationship between the entanglement-assisted and unassisted capacities,

$$C_E(\Phi) \leqslant C(\Phi) + \log d \tag{26}$$

where d is the dimension of the Hilbert space $\mathcal{H}_{in}$. This result can also be obtained from [19]. If the additivity of the classical capacity holds, we can replace $C(\Phi)$ by one-shot classical capacity $\chi^*(\Phi)$. Since Shor [14] already proved that the classical capacity of an entanglement breaking channel is additive, for entanglement breaking channel Φ, we have

$$C_E(\Phi) \leqslant \chi^*(\Phi) + \log d. \tag{27}$$

Next, we show that a tighter upper bound can be obtained for an entanglement breaking channel. Because Φ is an entanglement breaking channel, we have

$$(\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|) = \sum_i p_i \rho_A^i \otimes \rho_B^i \tag{28}$$

where both ρ_A^i and ρ_B^i are pure states. By strong concavity of von Neumann entropy, we know

$$S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)) \geqslant \begin{cases} S\big(\sum_i p_i \rho_A^i\big) + \sum_i p_i S\big(\rho_B^i\big) = S(\Phi(\rho_A)) \\ \sum_i p_i S\big(\rho_A^i\big) + S\big(\sum_i p_i \rho_B^i\big) = S(\rho_B) = S(\rho_A). \end{cases} \tag{29}$$

Substituting these relations into the BSST theorem (25), we have

$$C_E(\Phi) \leqslant \max_{\rho_A \in \mathcal{H}_{in}} S(\rho_A) \leqslant \log d \tag{30}$$

or

$$C_E(\Phi) \leqslant \max_{\rho_A \in \mathcal{H}_{in}} S(\Phi(\rho_A)) \leqslant \log d. \tag{31}$$

So, we know for an entanglement breaking channel, the entanglement-assisted classical capacity has an upper bound

$$C_E(\Phi) \leqslant \log d. \tag{32}$$

Comparing this relation with the general relation (27), we find that the term $\chi^*(\Phi)$ does not appear here though it is not always zero. So, we show there is an upper bound for $C_E(\Phi)$ when Φ is an entanglement breaking channel. It might be interpreted as showing, since the channel itself is entanglement-breaking, the prior entanglement may not help much in increasing the classical capacity.

6. Relationship between entanglement-assisted and one-shot unassisted capacities

As already pointed out in last section, Holevo [16] found the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the unassisted classical capacity as relation (26). If the classical channel capacity is additive, which is a long-standing conjecture, we have the inequality

$$C_E(\Phi) \leqslant \chi^*(\Phi) + \log d. \tag{33}$$

For an arbitrary quantum channel Φ, if this relation does not hold, this means $C(\Phi) > \chi^*(\Phi)$, thus the additivity conjecture of classical channel capacity does not hold. So, (33) may provide a criterion to test the additivity problem of classical capacity. Since the additivity of classical capacity is one of the most fundamental problems in the quantum information processing field, it should be examined whether this method really works or not. We show in this section that relation (33) always holds for an arbitrary quantum channel Φ. Thus it cannot provide a counterexample for additivity of classical capacity of quantum channels.

We assume that ρ_A have the following pure state decomposition:

$$\rho_A = \sum_j q_j \big|\tilde{\Psi}_A^j\big\rangle\big\langle\tilde{\Psi}_A^j\big|. \tag{34}$$

Using the same technique as that of [14], we define

$$|\tilde{\Psi}_{ABC}\rangle = \sum_j \sqrt{q_j}\big|\tilde{\Psi}_A^j\big\rangle|j\rangle_B|j\rangle_C. \tag{35}$$

So, we have

$$(\Phi \otimes I_{BC})(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|) = \sum_{jj'} \sqrt{q_j q_{j'}}\,\Phi\big(\big|\tilde{\Psi}_A^i\big\rangle\big\langle\tilde{\Psi}_A^{j'}\big|\big) \otimes |j\rangle_B\langle j'| \otimes |j\rangle_C\langle j'|. \tag{36}$$

With the help of the quantum entropy inequality [20], we obtain

$$\begin{aligned} S((\Phi \otimes I_{BC})(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|)) &\geqslant S((\Phi \otimes I_B)(\tilde{\rho}_{AB})) - S(\tilde{\rho}_C) \\ &= S\left(\sum_j q_j \Phi\big(\big|\tilde{\Psi}_A^j\big\rangle\big\langle\tilde{\Psi}_A^j\big|\big) \otimes |j\rangle_B\langle j|\right) - S\left(\sum_j q_j |j\rangle_C\langle j|\right) \\ &= \sum_j q_j S\big(\Phi\big(\big|\tilde{\Psi}_A^j\big\rangle\big\langle\tilde{\Psi}_A^j\big|\big)\big). \end{aligned} \tag{37}$$

We know

$$S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)) = S((\Phi \otimes I)(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|)) \tag{38}$$

where both $|\Psi_{AB}\rangle$ and $|\tilde{\Psi}_{ABC}\rangle$ are purifications of ρ_A. From BSST theorem (25), we have

$$\begin{aligned} C_E(\Phi) &= \max_{\rho_A \in \mathcal{H}_{in}} S(\rho_A) + S(\Phi(\rho_A)) - S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)) \\ &\leqslant \max_{\rho_A \in \mathcal{H}_{in}} S(\rho_A) + S(\Phi(\rho_A)) - \sum_j q_j S\big(\Phi\big(\big|\tilde{\Psi}_A^j\big\rangle\big\langle\tilde{\Psi}_A^j\big|\big)\big) \\ &\leqslant \log d + \chi^*(\Phi). \end{aligned} \tag{39}$$

Thus, we conclude, for an arbitrary quantum channel Φ, the entanglement-assisted and one-shot unassisted capacities have the relationship

$$C_E(\Phi) \leqslant \chi^*(\Phi) + \log d. \tag{40}$$

If the additivity of classical capacity holds, this relation is the same as relation (26). If the additivity does not hold for classical capacity, this relation is tighter than (26). This is the main result of this paper.

7. Summary

In summary, we pointed out that another quantum entropy inequality, the concavity of conditional entropy inequality, is equivalent to four other equivalent quantum entropy inequalities. Using directly the strong concavity of von Neumann entropy, the additivity of capacity of entanglement breaking channels can be proved simply. We also showed, for an entanglement breaking channel, that the entanglement-assisted channel capacity is upper bounded by $\log d$ which is tighter than the general case. A new upper bound is obtained for the entanglement-assisted classical capacity, the entanglement-assisted classical capacity is upper bounded by the sum of $\log d$ and the one-shot unassisted capacity. This result also eliminates one possible way to test the non-additivity of classical capacity.

Note added. Since this paper was posted, Holevo has presented another proof of (33) in [21].

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